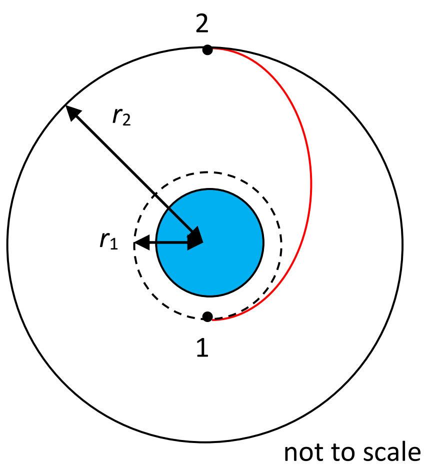
# Teacher notes

# Topic D

## The Hohmann transfer orbit

A satellite of mass $m$ is in a circular orbit of radius $r_1$. We wish to place the satellite in a higher circular orbit of radius $r_2$. One way for doing this is for the satellite to fire its engines at 1, increasing its speed and entering the red elliptical orbit. At 2 the engines are fired again increasing the speed again to the speed appropriate to the new circular orbit. This was described by Walter Hohmann in 1920.




The mass of the Earth is $M$.

(a) The speed of the satellite **in the elliptical orbit** at 1 is $v_1$ and at 2 it is $v_2$. Explain why $mv_1r_1 = mv_2r_2$.

(b) Apply conservation of total energy at 1 and 2 and the result in (a) to deduce that $v_1 = \sqrt{\dfrac{2GM}{r_1+r_2}\dfrac{r_2}{r_1}}$ and $v_2 = \sqrt{\dfrac{2GM}{r_1+r_2}\dfrac{r_1}{r_2}}$.

(c) Show that the work done by the engines of the satellite when they are fired at 1 is $W_2 = \dfrac{1}{2}\dfrac{GMm}{r_2}\dfrac{r_2-r_1}{r_2+r_1}$.

(d) Calculate the work done at 2.

(e) Hence show that total work done by the engines at 1 and at 2 is $\Delta E = \dfrac{1}{2}GMm\dfrac{r_2-r_1}{r_2r_1}$.

(f) Calculate the change in the total energy of the satellite in going from the circular orbit of radius $r_1$ to the circular orbit of radius $r_2$. Take a satellite mass of 500 kg.

(g) Compare the answers to (e) and (f).

(h) How long would it take to move a satellite from an orbit at a height of 300 km above the Earth's surface to another orbit at a height of $3.6\times10^7$ m? The mass of Earth is $6.0\times10^{24}$ kg and its radius is $6.4\times10^6$ m.

## Answers

(a) $mv_1r_1$ is the angular momentum of the satellite at 1 and $mv_2r_2$ is the angular momentum at 2. There are no external torques and so angular momentum is conserved. Hence $mv_1r_1 = mv_2r_2$.

(b) Conservation of energy says that $\frac{1}{2}mv_1^2 - \frac{GMm}{r_1} = \frac{1}{2}mv_2^2 - \frac{GMm}{r_2}$. Use $v_2 = v_1\frac{r_1}{r_2}$ from (a) to get

$\frac{1}{2}v_1^2 - \frac{GM}{r_1} = \frac{1}{2}v_1^2(\frac{r_1}{r_2})^2 - \frac{GM}{r_2}$ i.e.

$$v_1^2\left(1-(\frac{r_1}{r_2})^2\right) = \frac{2GM}{r_1} - \frac{2GM}{r_2}$$

$$v_1^2 = \frac{\frac{2GM}{r_1} - \frac{2GM}{r_2}}{1-(\frac{r_1}{r_2})^2} = \frac{\frac{2GM}{r_1r_2}(r_2-r_1)}{\frac{r_2^2-r_1^2}{r_2^2}} = \frac{\frac{2GM}{r_1r_2}(r_2-r_1)}{\frac{(r_2-r_1)(r_2+r_1)}{r_2^2}}$$

$$v_1 = \sqrt{\frac{2GM}{r_2+r_1}\frac{r_2}{r_1}}$$

Similarly, $v_2 = \sqrt{\frac{2GM}{r_2+r_1}\frac{r_1}{r_2}}$.

(c) At 1 when in circular orbit the speed is $v_1 = \sqrt{\frac{GM}{r_1}}$. The work done by the net force is the change in kinetic energy. The forces on the satellite are the gravitational force and the engine force. The gravitational force is normal to the path and so does zero work. The total work is thus done by the engines, which is what we want to calculate. So

$$W_1 = \frac{1}{2}m\left(\frac{2GM}{r_2+r_1}\frac{r_2}{r_1} - \frac{GM}{r_1}\right)$$

$$= \frac{1}{2}\frac{GMm}{r_1}\left(\frac{2r_2}{r_2+r_1} - 1\right)$$

$$= \frac{1}{2}\frac{GMm}{r_1}\frac{r_2-r_1}{r_2+r_1}$$

(d) In the same way, $W_2 = \frac{1}{2}\frac{GMm}{r_2}\frac{r_2-r_1}{r_2+r_1}$.

(e) The total work done by the engines is then

$$W_1 + W_2 = \frac{1}{2}\frac{GMm}{r_1}\frac{r_2 - r_1}{r_2 + r_1} + \frac{1}{2}\frac{GMm}{r_2}\frac{r_2 - r_1}{r_2 + r_1}$$
$$= \frac{1}{2}GMm\frac{r_2 - r_1}{r_2 + r_1}(\frac{1}{r_2} + \frac{1}{r_1})$$
$$= \frac{1}{2}GMm\frac{r_2 - r_1}{r_2 + r_1}\frac{r_2 + r_1}{r_2 r_1}$$
$$= \frac{1}{2}GMm\frac{r_2 - r_1}{r_2 r_1}$$

As we will see in (f) this is the same as the change in total energy from one orbit to the other.

(f) The change in total energy in moving from one orbit to the other is

$$\Delta E = -\frac{GMm}{2r_2} - (-\frac{GMm}{2r_1})$$
$$= \frac{GMm}{2}(\frac{1}{r_1} - \frac{1}{r_2})$$
$$= \frac{1}{2}GMm\frac{r_2 - r_1}{r_2 r_1}$$

The two orbit radii are $r_1 = 6.4 \times 10^6 + 300 \times 10^3 = 6.7 \times 10^6$ m and $r_2 = 6.4 \times 10^6 + 3.6 \times 10^7 = 4.2 \times 10^7$ m. Numerically, the energy difference is

$$\Delta E = \frac{1}{2} \times 6.67 \times 10^{-11} \times 6.0 \times 10^{24} \times 500 \times \frac{4.2 \times 10^7 - 6.7 \times 10^6}{4.2 \times 10^7 \times 6.7 \times 10^6} = 1.3 \times 10^{10} \text{ J}.$$

(g) The change in energy in (f) is of course the same as the total work done by the engines. This is because the change in the total energy is the work done by the **external** forces i.e. the engines.

(h) The two orbit radii are $r_1 = 6.7 \times 10^6$ m and $r_2 = 4.2 \times 10^7$ m. The semi-major axis of the ellipse is then $a = \frac{6.7 \times 10^6 + 4.2 \times 10^7}{2} = 2.5 \times 10^7$ m. We need half the ellipse revolution period and by Kepler's third law, $\frac{1}{2}T = \frac{1}{2}\sqrt{\frac{4\pi^2}{GM}a^3} = \frac{1}{2}\sqrt{\frac{4\pi^2 \times (2.5 \times 10^7)^3}{6.67 \times 10^{-11} \times 6.0 \times 10^{24}}} \approx 2 \times 10^4 \text{ s} \approx 5.5 \text{ hours}$.